Lattice systems:
The derivation of hydrodynamic equations

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Description of problems

Consider a dynamical system

$$\dot{Y}(t) = \mathcal{F}(Y(t)), \quad t \in \mathbb{R}; \quad Y(0) = Y_0.$$

$$Y_0 \in \mathcal{H} - \text{ phase space}$$

$$U(t) : Y_0 \to Y(t), \quad t \in \mathbb{R}.$$

Problem 1. Let μ_0 be a Borel probability measure on $\mathcal{H}$.

Def. $\{\mu_t, t \in \mathbb{R}\}$: $\mu_t(B) = \mu_0(U^{-1}(t)B)$, $B \in \mathcal{B}(\mathcal{H})$.

Theorem $\boxed{\mu_t \to \mu_\infty \text{ as } t \to \infty\,.}$

Problem 2. Let $\{\mu_0^\varepsilon, \varepsilon > 0\}$ be a family of initial measures on $\mathcal{H}$.
"*Time-space scaling*": $t \to \tau/\varepsilon$, $x \to r/\varepsilon$.

Def. For $\tau \neq 0$, $r \in \mathbb{R}^d$, $\varepsilon > 0$, $\mu^\varepsilon_{\tau/\varepsilon, r/\varepsilon}$ – distribution of the random solution $Y(r/\varepsilon, \tau/\varepsilon)$.

Theorem $\boxed{\text{For } \tau \neq 0\,,\ r \in \mathbb{R}^d,\ \mu^\varepsilon_{\tau/\varepsilon, r/\varepsilon} \to \mu_{\tau,r} \text{ as } \varepsilon \to 0\,.}$

Corollary The correlation functions of limit measures $\mu_{\tau,r}$ satisfy the Euler type equation.

1. Model: Harmonic Crystals [1]

We study the dynamics of the harmonic crystals in $\mathbb{Z}^d$, $d \geq 1$,

$$\ddot{u}(z,t) = -\sum_{z' \in \mathbb{Z}^d} V(z-z')u(z',t), \quad z \in \mathbb{Z}^d, \quad t \in \mathbb{R}, \tag{1}$$

$$u(z,0) = u_0(z), \quad \dot{u}(z,0) = u_1(z), \quad z \in \mathbb{Z}^d. \tag{2}$$

$V(z)$ is the interaction (or force) matrix, $(V_{kl}(z))_{k,l=1}^n$,
$u(z,t) = (u_1(z,t), \ldots, u_n(z,t)) \in \mathbb{R}^n$, $n \geq 1$.

Hamiltonian functional

$$H(u_0(\cdot), u_1(\cdot)) = \frac{1}{2}\sum_{z \in \mathbb{Z}^d} |u_1(z)|^2 + \frac{1}{2}\sum_{z,z' \in \mathbb{Z}^d} u_0(z) \cdot V(z-z')u_0(z')$$

Example: *the nearest neighbor crystal or the simple elastic lattice:*

$$V_{kl}(z) = 0, \quad k \neq l, \quad V_{kk}(z) = \begin{cases} 2d\gamma_k + m_k^2 & \text{for } z = 0 \\ -\gamma_k & \text{for } |z| = 1 \\ 0 & \text{for } |z| \geq 2 \end{cases}$$

with $\gamma_k > 0$, $m_k \geq 0$. In this case, Eq.(1) becomes

$$\ddot{u}_k(z,t) = (\gamma_k \Delta_L - m_k^2)u_k(z,t), \quad k = 1, \ldots, n,$$

Δ_L – the discrete Laplace operator on the lattice $\mathbb{Z}^d$,

$$\Delta_L v(z) := \sum_{j=1}^{d}(v(z+e_j) - 2v(z) + v(z-e_j)), \quad e_j = (\delta_{1j}, ..., \delta_{dj}).$$

[1] see A.A. Maradudin, E.W. Montroll, G.H. Weiss (with I.P. Ipatova), *Theory of Lattice Dyamics in the Harmonic Approximation*, 1963.
O.E. Lanford III, J.L. Lebowitz, *Time Evolution and Ergodic Properties of Harmonic Systems*, in: *Dynamical Systems, Theory and Applications*, Lecture Notes in Physics **38** (1975).

Conditions on the interaction matrix V

V1 $\exists\, C, \gamma > 0$: $\|V(z)\| \le Ce^{-\gamma|z|}$, $z \in \mathbb{Z}^d$.

V2 $V(z)$ is real and symmetric: $V^T(-z) = V(z) \in \mathbb{R}$

$$\hat{V}(\theta) := \sum_{z\in\mathbb{Z}^d} V(z)e^{iz\cdot\theta}, \ \theta \in \mathbb{T}^d \ (\mathbb{T}^d \text{ -- } d\text{-torus } \mathbb{R}^d/(2\pi\mathbb{Z})^d)$$

V1, **V2** $\Longrightarrow$ $\hat{V}(\theta) = \hat{V}^*(\theta)$, real-analytic.

V3 $\hat{V}(\theta) \ge 0$ for every $\theta \in \mathbb{T}^d$.

Def. $\Omega(\theta) := (\hat{V}(\theta))^{1/2} \ge 0$

$\omega_\sigma(\theta)$, $\sigma = 1, ..., s$, – eigenvalues of $\Omega(\theta)$ (*dispersion relations*),

$0 \le \omega_1(\theta) < \omega_2(\theta) < \ldots < \omega_s(\theta)$ $(1 \le s \le n)$

Lemma *(i)* $\omega_\sigma(\theta)$ *– real-analytic for* $\theta \in \mathbb{T}^d \setminus \mathcal{C}_*$ *with* $mes\, \mathcal{C}_* = 0$. *(ii)* $\Omega(\theta) = \sum_{\sigma=1}^{s} \omega_\sigma(\theta)\Pi_\sigma(\theta)$, *where spectral projections* $\Pi_\sigma(\theta)$ *– real-analytic on* $\mathbb{T}^d \setminus \mathcal{C}_*$.

V4 $\det\Big(\dfrac{\partial^2}{\partial\theta_i\partial\theta_j}\omega_\sigma(\theta)\Big) \not\equiv 0$, $\theta \in \mathbb{T}^d \setminus \mathcal{C}_*$, $\sigma = 1, \ldots, s$.

V5 For each $\sigma \ne \sigma'$, $\omega_\sigma(\theta) \pm \omega_{\sigma'}(\theta) \not\equiv \text{const}_\pm$, with $\text{const}_\pm \ne 0$.

V6 $\|\hat{V}^{-1}(\theta)\| \in L^1(\mathbb{T}^d)$.

Remark In the case of *the nearest neighbor crystal*, the eigenvalues of $\Omega(\theta)$ are

$$\omega_k(\theta) = \sqrt{2\gamma_k(1-\cos\theta_1) + \ldots + 2\gamma_k(1-\cos\theta_d) + m_k^2}, \ k = 1, .., n.$$

Condition **V6** $\iff$ $\omega_k^{-2}(\theta) \in L^1(\mathbb{T}^d)$. Hence, **V1**–**V6** hold if either (i) $d \geq 3$ or (ii) $d = 1, 2$ and $m_k > 0 \ \ \forall k$.

Write $Y(t) = (Y^0(t), Y^1(t)) \equiv (u(\cdot, t), \dot{u}(\cdot, t))$,

$Y_0 = (Y_0^0, Y_0^1) \equiv (u_0(\cdot), u_1(\cdot))$.

Then (1)–(2) becomes the evolution equation

$$\dot{Y}(t) = \mathcal{A}Y(t), \quad t \in \mathbb{R}, \quad Y(0) = Y_0, \tag{3}$$

where $\mathcal{A} = \begin{pmatrix} 0 & 1 \\ -\mathcal{V} & 0 \end{pmatrix}$, $\mathcal{V}u = \sum\limits_{z' \in \mathbb{Z}^d} V(z - z')u(z')$.

<u>Def.</u> $\mathcal{H}_\alpha$, $\alpha \in \mathbb{R}$, is the Hilbert space of $Y_0 = (u_0, u_1) \in \mathbb{R}^n \times \mathbb{R}^n$, $\|Y_0\|_\alpha^2 = \sum\limits_{z \in \mathbb{Z}^d} |Y_0(z)|^2(1 + |z|^2)^\alpha < \infty.$

<u>Lemma</u> **V1**, **V2** $\Longrightarrow$ $\forall\, Y_0 \in \mathcal{H}_\alpha$ $\exists!\, Y(t) = U(t)Y_0 \in C(\mathbb{R}, \mathcal{H}_\alpha)$ to the problem (3).

2. Convergence to Equilibrium

Random initial data $Y_0 = (Y_0^0, Y_0^1) \equiv (u_0(\cdot), u_1(\cdot))$
μ_0 is a Borel probability measure on $\mathcal{H}_\alpha$ giving the distribution of Y_0.

Conditions on the initial measure μ_0

S1 $\mathbb{E}_0(Y_0(z)) \equiv \int Y_0(z)\,\mu_0(dY_0) = 0\,,\ z \in \mathbb{Z}^d.$

S2 $Q_0(z,z') = \mathbb{E}_0(Y_0(z) \otimes Y_0(z')) = q_0(z-z')\,,\ z,z' \in \mathbb{Z}^d.$

S3 μ_0 has finite mean energy density,

$$\mathbb{E}_0(|Y_0^0(z)|^2 + |Y_0^1(z)|^2) = \mathrm{tr}\ \left[Q_0^{00}(z,z) + Q_0^{11}(z,z)\right] \le e_0 < \infty.$$

Def. $\sigma(\mathcal{A}) = \sigma\{Y_0(z),\ z \in \mathcal{A}\}\,,\ \mathcal{A} \subset \mathbb{Z}^d.$
The Ibragimov mixing coefficient of μ_0 on $\mathcal{H}_\alpha$:

$$\varphi(r) = \sup_{\substack{\mathcal{A},\mathcal{B} \subset \mathbb{Z}^d \\ \mathrm{dist}(\mathcal{A},\mathcal{B}) \ge r}} \ \sup_{\substack{A \in \sigma(\mathcal{A}),\, B \in \sigma(\mathcal{B}) \\ \mu_0(B) > 0}} \frac{|\mu_0(A \cap B) - \mu_0(A)\mu_0(B)|}{\mu_0(B)}.$$

S4 μ_0 satisfies the *strong uniform* Ibragimov mixing condition[2],

i.e., $\varphi(r) \to 0$ as $r \to \infty$. Moreover, $\displaystyle\int_0^\infty r^{d-1}\varphi^{1/2}(r)\,dr < \infty$.

Remark Instead of the *strong uniform* Ibragimov mixing condition, it suffices to assume the *uniform* Rosenblatt mixing condition [3] together with a higher power > 2 in the bound **S4**. The uniform Rosenblatt mixing condition also could be weakened.

[2] I.A. Ibragimov, Yu.V. Linnik, *Independent and Stationary Sequences of Random Variables*, 1971.
[3] M.A. Rosenblatt, A central limit theorem and a strong mixing condition, *Proc. Nat. Acad. Sci. U.S.A.* **42** (1956), 43-47.

Def. μ_t is the Borel probability measure on $\mathcal{H}_\alpha$ giving the distribution of the random solution $Y(t)$:

$$\mu_t(B) = \mu_0(U(-t)B), \quad B \in \mathcal{B}(\mathcal{H}_\alpha), \quad t \in \mathbb{R}.$$

Theorem 1 Let $\alpha < -d/2$. Then

(i) $\mu_t \overset{\mathcal{H}_\alpha}{\rightharpoondown} \mu_\infty$, $t \to \infty$.

This means $\lim\limits_{t\to\infty} \int f(Y)\,\mu_t(dY) = \int f(Y)\,\mu_\infty(dY)$, $\forall f \in C_b(\mathcal{H}_\alpha)$.

(ii) μ_∞ is Gaussian translation-invariant measure on $\mathcal{H}_\alpha$.

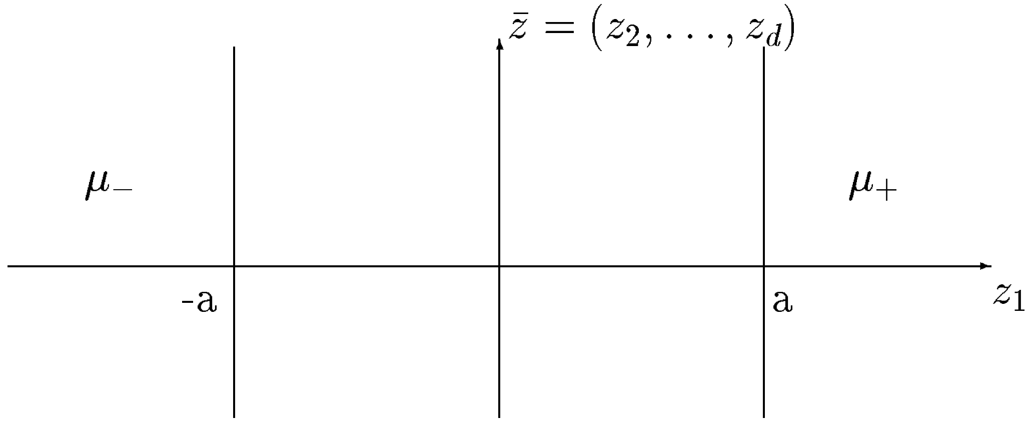
(iii) The correlation matrices of the measures μ_t converge to a limit,

$$Q_t(z,z') = \int (Y(z) \otimes Y(z'))\,\mu_t(dY) \to q_\infty(z-z'), \quad t \to \infty.$$

Theorem 1 is proved in [T.D., A. Komech, H. Spohn, *J. Math. Phys.* **44** (2003), 2596-2620].

ArXiv: math-ph/0210039

Non translation-invariant initial measures μ_0



$$Q_0(z, z') = \begin{cases} q_-(z - z'), & z_1, z_1' < -a, \\ q_+(z - z'), & z_1, z_1' > a, \end{cases}$$

where $q_\pm(z)$ – correlation functions of some translation-invariant measures $\mu_\pm$ on $\mathcal{H}_\alpha$ with zero mean.

Theorem 2 (see [4]) $\mu_t \to \mu_\infty$, $t \to \infty$, on the space $\mathcal{H}_\alpha$, $\alpha < -d/2$.

Application: Let $\mu_\pm = g_\pm$ be Gibbs measures, corresponding to positive temperatures $T_- \neq T_+$. Formally,

$$g_\pm(dY_0) = \frac{1}{Z_\pm} e^{-\frac{\beta_\pm}{2} H(Y_0)} \prod_z dY_0(z), \quad \beta_\pm = T_\pm^{-1}.$$

Def. $g_\pm$ are Gaussian measures with correlation functions $q_\pm(z - z')$ of the form $q_\pm(z) = F^{-1}_{\theta \to z}[\hat{q}_\pm(\theta)]$, where $\hat{q}_\pm(\theta) = T_\pm \begin{pmatrix} \hat{V}^{-1}(\theta) & 0 \\ 0 & I \end{pmatrix}$.

Lemma The limiting mean density of current energy is

$$\mathbf{j}_\infty = -C(T_+ - T_-, 0, \ldots, 0), \quad \text{with constant } C > 0,$$

that corresponds to Second law of thermodynamics.

[4] Theorem 2 is proved (i) for $d = 1$ in [C. Boldrighini, A. Pellegrinotti, L. Triolo, *J. Stat. Phys.*, **30** (1983), 123-155]; (ii) for $d \geq 1$ - in [T.D., A. Komech, N. Mauser, *J. Stat. Phys.* **114** (2004), no.3/4, 1035-1083. ArXiv: math-ph/0211017]

Remark Convergence to equilibrium measure was proved also for continuous systems describing by partial differential equations (see [5]):

I. Wave equations

$$\ddot{u}(x,t)=\sum_{i,j=1}^{d}\partial_i(a_{ij}(x)\partial_j u(x,t))-a_0(x)\,u(x,t),\quad x\in\mathbb{R}^d,\quad t\in\mathbb{R},$$

$$a_{ij}(x)=\delta_{ij}+b_{ij}(x)\,,\ b_{ij},a_0\in C_0^\infty(\mathbb{R}^d)\,,\ a_0(x)\geq 0\,;$$

II. Klein-Gordon equations

$$\ddot{u}(x,t)=\sum_{j=1}^{d}(\partial_j-iA_j(x))^2u(x,t)-m^2\,u(x,t),\quad x\in\mathbb{R}^d,\quad t\in\mathbb{R},$$

$$m>0\,,\ A_j\in C_0^\infty(\mathbb{R}^d)\,,\ \partial_2A_1\not\equiv\partial_1A_2\ \text{ for }\ d=2\,;$$

III. Hamiltonian system consisting of the scalar Klein-Gordon field coupled to the "simple lattice"

$$\left\{\begin{array}{rcl}\dot{\psi}(x,t)&=&(\Delta-m_0^2)\psi(x,t)-\sum\limits_{k'\in\mathbb{Z}^d}u(k',t)\cdot R(x-k'),\quad x\in\mathbb{R}^d,\\ \dot{u}(k,t)&=&(\Delta_L-\nu_0^2)u(k,t)-\int R(x'-k)\psi(x',t)\,dx',\quad k\in\mathbb{Z}^d,\end{array}\right.$$

$$m_0,\nu_0>0\,,\ R\in C^\infty(\mathbb{R}^d)\,,\ \exists\delta>0:\ |R(x)|\leq Ce^{-\delta|x|}\,;$$

IV. Hamiltonian system consisting of a real-valued vector field $\psi=(\psi_1,\ldots,\psi_n)$ and a particle with position $q\in\mathbb{R}^3$

$$\left\{\begin{array}{rcl}\dot{\psi}_k(x,t)&=&(\Delta-m_k^2)\psi_k(x,t)-q(t)\cdot\nabla\rho_k(x),\quad x\in\mathbb{R}^3,\quad t\in\mathbb{R},\\ \dot{q}(t)&=&-\omega^2q(t)-\sum\limits_{k=1}^{n}\int\psi_k(x,t)\nabla\rho_k(x)\,dx,\end{array}\right.$$

$$m_k\geq 0\,,\ \omega>0\,,\ \rho_k\in C_0^\infty(\mathbb{R}^3)\,,\ k=1,\ldots,n\,.$$

[5]**I**: ($d\geq 3$, *odd*) N. Ratanov (1984); T.D., A. Komech, H. Spohn (2002) ArXiv: math-ph/0508044; ($d\geq 4$, *even*) T.D. (2005)
II: E. Kopylova (1986); T.D., A. Komech, *Theory Probab. Appl.* **50** (2006), no.4, 582-611
III: T.D., A. Komech, *Russian J. Math. Phys.* **12** (2005), no.3, 301-325. ArXiv: math-ph/0508053
IV: T.D., *Russian J. Math. Phys.* **17** (2010), no.1, 77-95. ArXiv: 0711.1091

3. Derivation of Limiting "Hydrodynamic" Eq.

Let $\varepsilon > 0$ be a small scale parameter,
$\{\mu_0^\varepsilon, \varepsilon > 0\}$ be a family of the initial measures.

3.1. Conditions on μ_0^ε

$$\mathbb{E}_0^\varepsilon(Y_0(z)) \equiv \int Y_0(z)\, \mu_0^\varepsilon(dY_0) = 0$$

$$Q_\varepsilon(z, z') := \mathbb{E}_0^\varepsilon(Y(z) \otimes Y(z')), \quad z, z' \in \mathbb{Z}^d.$$

R1 $\forall \varepsilon > 0$, $z, z' \in \mathbb{Z}^d$,

$$|Q_\varepsilon(z, z')| \le C(1 + |z - z'|)^{-\gamma}, \quad \gamma > d.$$

R2 For any $r \in \mathbb{R}^d$, $z, z' \in \mathbb{Z}^d$,

$$Q_\varepsilon([r/\varepsilon] + z, [r/\varepsilon] + z') \to R(r, z - z') \quad \text{as} \quad \varepsilon \to 0.$$

Measures μ_0^ε are locally homogeneous for space translations of order much less than ε^{-1}, and nonhomogeneous for translations $\sim \varepsilon^{-1}$.

3.2. Weak convergence

For $\tau \neq 0$, $r \in \mathbb{R}^d$, let us consider the random field $Y(z,t)$:

$$\boxed{z \sim [r/\varepsilon] \in \mathbb{Z}^d,\ t = \tau/\varepsilon^\kappa,\ \kappa > 0}$$

Def. (i) μ_t^ε – the distribution of the random solution $Y(t)$,

$$\mu_t^\varepsilon(B) = \mu_0^\varepsilon(U(-t)B), \quad B \in \mathcal{B}(\mathcal{H}_\alpha), \quad t \in \mathbb{R}.$$

(ii) For $\tau \neq 0$, $r \in \mathbb{R}^d$, $\mu^\varepsilon_{\tau/\varepsilon^\kappa, r/\varepsilon}(B) := \mu^\varepsilon_{\tau/\varepsilon^\kappa}(T_{[r/\varepsilon]}B)$, $B \in \mathcal{B}(\mathcal{H}_\alpha)$,
T_h – the group of space translations: $T_h Y_0(z) = Y_0(z-h)$, $h \in \mathbb{Z}^d$.

Theorem 3 Let $\tau \neq 0$, $r \in \mathbb{R}^d$, $\alpha < -d/2$.

(i) Let $\kappa < 1$. Then $\mu^\varepsilon_{\tau/\varepsilon^\kappa, r/\varepsilon} \overset{\mathcal{H}_\alpha}{\longrightarrow} \mu^G_r$, $\varepsilon \to 0$ (in the sense of weak convergence).

(ii) Let $\kappa = 1$. Then $\mu^\varepsilon_{\tau/\varepsilon, r/\varepsilon} \overset{\mathcal{H}_\alpha}{\longrightarrow} \mu^G_{\tau,r}, \quad \varepsilon \to 0.$

$\mu^G_{\tau,r}$ is a Gaussian measure on $\mathcal{H}_\alpha$, which is invariant under the time translation $U(t)$.

3.3. Euler limit ($\kappa = 1$)

Theorem 4 The correlation functions of $\mu^{\varepsilon}_{\tau/\varepsilon, r/\varepsilon}$ converge to a limit,

$$\lim_{\varepsilon\to 0} Q_{\varepsilon,\tau/\varepsilon}([r/\varepsilon]+z, [r/\varepsilon]+z') = q_{\tau,r}(z-z'), \quad z, z' \in \mathbb{Z}^d.$$

Remark $\hat{q}_{\tau,r}(\theta) = (\hat{q}^{ij}_{\tau,r}(\theta))^1_{i,j=0}$ satisfies the equilibrium condition, $\hat{q}^{11}_{\tau,r}(\theta) = \hat{V}(\theta)\hat{q}^{00}_{\tau,r}(\theta)$, $\hat{q}^{01}_{\tau,r}(\theta) = -\hat{q}^{10}_{\tau,r}(\theta)$.

Corollary Let $r \in \mathbb{R}^d$, $\tau \neq 0$. In the σ -band the matrix-valued function $\hat{q}_{\tau,r}(\theta)$ satisfies the *"hydrodynamic" Euler type equation*:

$$\begin{cases} \partial_\tau f_\sigma(\tau, r; \theta) &= iC_\sigma(\theta)\nabla\omega_\sigma(\theta)\cdot\nabla_r f_\sigma(\tau, r; \theta), \;\; r \in \mathbb{R}^d, \;\; \tau > 0, \\ \\ f_\sigma(\tau, r, \theta)|_{\tau=0} &= \dfrac{1}{2}\Pi_\sigma(\theta)\left(\hat{R}(r,\theta) + C_\sigma(\theta)\hat{R}(r,\theta)C^*_\sigma(\theta)\right)\Pi_\sigma(\theta), \end{cases}$$

where $C_\sigma = \begin{pmatrix} 0 & \omega_\sigma^{-1} \\ -\omega_\sigma & 0 \end{pmatrix}$, $\Pi_\sigma(\theta)$ – spectral projections.

3.4. Wigner matrices

Def. (i) $a(z) := \frac{1}{\sqrt{2}}(\mathcal{V}^{1/4}u_0(z) + i\,\mathcal{V}^{-1/4}u_1(z)) \in \mathbb{C}^n$, $z \in \mathbb{Z}^d$,

where $\mathcal{V}u = \sum\limits_{z' \in \mathbb{Z}^d} V(z-z')u(z')$.

(ii) The scaled $n \times n$ Wigner matrix is

$$W^\varepsilon(\tau; r, \theta) = \sum_{y \in \mathbb{Z}^d} e^{i\theta\cdot y}\, \mathbb{E}^\varepsilon_{\tau/\varepsilon}(a^*([r/\varepsilon + y/2]) \otimes a([r/\varepsilon - y/2])).$$

Conditions on $Q_\varepsilon(z,z') \implies \exists \lim\limits_{\varepsilon\to 0} W^\varepsilon(0; r,\theta) =: W_0(0; r, \theta)$.

Theorem 5 For any $r \in \mathbb{R}^d$, $\tau > 0$,

$$\lim_{\varepsilon\to 0} W^\varepsilon(\tau; r, \theta) = W^p(\tau; r, \theta) \quad \text{(in the sense of distributions)},$$

where the limit Wigner matrix is

$$W^p(\tau; r, \theta) = \sum_{\sigma=1}^{s} \Pi_\sigma(\theta) W_0(0; r - \tau\nabla\omega_\sigma(\theta), \theta)\Pi_\sigma(\theta).$$

Corollary Denote by $W^p_\sigma(\tau; r, \theta)$, $\sigma = 1, \ldots, s$, the σ-th band of $W^p(\tau; r, \theta)$. Then W^p_σ is a solution of **energy transport equation**

$$\begin{cases} \partial_\tau W^p_\sigma(\tau; r,\theta) + \nabla\omega_\sigma(\theta)\cdot\nabla_r W^p_\sigma(\tau; r,\theta) = 0, \quad \tau > 0, \quad r \in \mathbb{R}^d, \\[2ex] W^p_\sigma(\tau; r, \theta)|_{\tau=0} = \Pi_\sigma(\theta) W_0(0; r, \theta)\Pi_\sigma(\theta), \quad r \in \mathbb{R}^d. \end{cases}$$

For $d = 1$, Theorem 4 was proved in [R.L. Dobrushin, A. Pellegrinotti, Yu.M. Suhov, L. Triolo, *J. Stat. Phys.* **43** (1986), 571-607.]

For $d \geq 1$, Theorems 3–5 are proved in [T.D., H. Spohn, *Markov Processes and Related Fields* **12** (2006), no.4, 645-578. ArXiv: math-ph/0505031]

4. Navier–Stokes Limit ($\kappa = 2$)

Random solution $Y(z,t)$: $\boxed{z \sim [r/\varepsilon],\ t = \tau/\varepsilon^2.}$

Theorem 6 Let $r \in \mathbb{R}^d$, $\tau \neq 0$. Then for any $z, z' \in \mathbb{Z}^d$, correlation functions of $\mu^{\varepsilon}_{\tau/\varepsilon^2, r/\varepsilon}$ have the following asymptotics

$$\lim_{\varepsilon\to 0}\Big(Q_{\varepsilon,\tau/\varepsilon^2}([r/\varepsilon]+z,[r/\varepsilon]+z') - F_\varepsilon(\tau, r, z-z')\Big) = 0.$$

The explicit formulas for $F_\varepsilon(\tau, r, z)$ are derived.

Corollary Set $f_\varepsilon(t,r,\theta) := \hat{F}_\varepsilon(\varepsilon t, r, \theta)$.

In the σ-th band ($\sigma = 1,\dots,s$) the function $f_\varepsilon \equiv f_\varepsilon(t,r,\theta)$ evolves according to the Navier–Stokes type equation

$$\begin{cases} \partial_t f_\varepsilon = iC_\sigma(\theta)\left(\nabla\omega_\sigma(\theta)\cdot\nabla_r f_\varepsilon + \dfrac{i\varepsilon}{2}\mathrm{tr}\left[\nabla^2\omega_\sigma(\theta)\cdot\nabla_r^2 f_\varepsilon\right]\right), \quad t>0, \\[2ex] f|_{t=0} = \dfrac{1}{2}\Pi_\sigma(\theta)\left(\hat{R}(r,\theta) + C_\sigma(\theta)\hat{R}(r,\theta)C^*_\sigma(\theta)\right)\Pi_\sigma(\theta), \quad r\in\mathbb{R}^d, \end{cases}$$

where $\nabla^2 f = \left(\dfrac{\partial^2 f}{\partial r_i \partial r_j}\right)_{i,j=1}^{d}$, $C_\sigma = \begin{pmatrix} 0 & \omega_\sigma^{-1} \\ -\omega_\sigma & 0 \end{pmatrix}$, $\sigma = 1,\dots,s$.

Theorem 6 is proved for $d=1$ in [R.L. Dobrushin, A. Pellegrinotti, Yu.M. Suhov, L. Triolo, *J. Stat. Phys.* **52** (1988), 423-439]; for $d \geq 1$ – in [T.D., accepted in *Theor. Math. Physics* (2011)]

5. Corrections of Higher Order ($\kappa > 2$)

Random solution $Y(z,t)$: $\boxed{z \sim [r/\varepsilon]\,,\ t = \tau/\varepsilon^{\kappa}\,,\ \kappa > 2\,.}$

<u>**Theorem 7**</u> Let $\kappa > 2$, $r \in \mathbb{R}^d$, $\tau > 0$. Then $\forall z, z' \in \mathbb{Z}^d$, correlation functions of $\mu^{\varepsilon}_{\tau/\varepsilon^{\kappa}, r/\varepsilon}$ have the following asymptotics

$$\lim_{\varepsilon \to 0} \Big(Q_{\varepsilon, \tau/\varepsilon^{\kappa}}([r/\varepsilon] + z, [r/\varepsilon] + z') - F_{\varepsilon}^{[\kappa]}(\tau, r, z - z') \Big) = 0.$$

The explicit formulas for $F_{\varepsilon}^{[\kappa]}(\tau, r, z)$ are derived.

<u>**Corollary**</u> Set $\tau = \varepsilon^{k-1} t$, $k \geq 2$. Then the matrix $\Pi_{\sigma}(\theta) \hat{F}_{\varepsilon}^{k}(\varepsilon^{k-1} t, r; \theta) \Pi_{\sigma}(\theta)$ (denote it by $f_{\sigma}^{\varepsilon}(t, r; \theta)$) is a solution of the following equation

$$\begin{aligned} \partial_t f_{\sigma}^{\varepsilon}(t, r; \theta) &= iC_{\sigma}(\theta) \Big(\nabla \omega_{\sigma}(\theta) \cdot \nabla_r + \frac{i\varepsilon}{2} \nabla^2 \omega_{\sigma}(\theta) \cdot \nabla_r^2 + \ldots \\ &\quad + \frac{(i\varepsilon)^{k-1}}{k!} \nabla^k \omega_{\sigma}(\theta) \cdot \nabla_r^k \Big) f_{\sigma}^{\varepsilon}(t, r; \theta), \ \ t > 0, \ \ r \in \mathbb{R}^d, \end{aligned}$$

where, by definition,

$$\nabla^k \omega_{\sigma}(\theta) \cdot \nabla_r^k f(r) := \sum_{i_1, \ldots, i_k = 1}^{d} \frac{\partial^k \omega_{\sigma}(\theta)}{\partial \theta_{i_1} \ldots \partial \theta_{i_k}} \frac{\partial^k f}{\partial r_{i_1} \ldots \partial r_{i_k}}, \qquad k \in \mathbb{N}.$$

Theorem 7 is proved in [T.D., accepted in *Theor. Math. Physics* (2011)], <u>Arxiv: 1110.0616 [math-ph]</u>

6. Harmonic Crystals in the Half-Space

$\mathbb{Z}^d_+ = \{z \in \mathbb{Z}^d : \ z_1 > 0\}$

We study the dynamics of the harmonic crystals in $\mathbb{Z}^d_+$, $d \geq 1$,

$$\begin{cases} \ddot{u}(z,t) = -\sum\limits_{z' \in \mathbb{Z}^d_+} \big(V(z-z') - V(z-\tilde{z}')\big)\, u(z',t), \quad z \in \mathbb{Z}^d_+, \quad t \in \mathbb{R}, \\ u(z,t)|_{z_1=0} = 0, \quad t \in \mathbb{R}, \\ u(z,0) = u_0(z), \quad \dot{u}(z,0) = u_1(z), \quad z \in \mathbb{Z}^d_+. \end{cases}$$

Here $\tilde{z} = (-z_1, \bar{z})$, $\bar{z} = (z_2, \ldots, z_d)$.

The matrix V satisfies conditions **V1**–**V6**. In addition,

$$V(z_1, z_2, \ldots) = V(-z_1, z_2, \ldots).$$

The results of all theorems remain true with some modifications [6]. In particular, the σ-th band of the Wigner function $W^p(\tau; r, \theta)$ (denote it by W^p_σ) is a solution of the following problem

$$\begin{cases} \partial_\tau W^p_\sigma(\tau; r, \theta) + \nabla\omega_\sigma(\theta) \cdot \nabla_r W^p_\sigma(\tau; r, \theta) = 0, \quad \tau > 0, \quad r \in \mathbb{R}^d_+, \\ W^p_\sigma(\tau; r, \theta)|_{\tau=0} = W_\sigma(0; r, \theta), \quad r \in \mathbb{R}^d_+, \\ W^p_\sigma(\tau; r, \theta)|_{r_1=0} = W_\sigma(0; \tau|\partial_1\omega_\sigma(\theta)|, \bar{r} - \tau\partial_{\bar{\theta}}\,\omega_\sigma(\theta), \theta), \quad \tau > 0. \end{cases}$$

[6] see [T.D., *Rus. J. Math. Phys.* **15** (2008), no.4, 460-472] ArXiv: 0905.3472;
[T.D., *J. Math. Phys.* **51** (2010)] ArXiv: 0905.4806