Decay of Entangled Photon Correlations in Hollow Waveguides and Fibres

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Börje Nilsson

Quantization of the electromagnetic field

Quantize the classical equations with spatiotemporal description.

The free electromagnetic field

Classically: Maxwell's equations with solution

$$\mathbf{B} = \nabla \times \mathbf{A},\ \ \mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t},\ \ \nabla \cdot \mathbf{A} = 0$$

$$\mathbf{A}(\mathbf{r}, t) = \int \frac{\mathrm{d}^3 k}{\sqrt{2\omega(k)}} \sum_{s=1}^{2} \varepsilon_s \left[a_s^+(\mathbf{k}) \mathrm{e}^{\mathrm{i}\omega(\mathbf{k})t - \mathrm{i}\mathbf{k}\cdot\mathbf{r}} + a_s(\mathbf{k}) \mathrm{e}^{-\mathrm{i}\omega(\mathbf{k})t + \mathrm{i}\mathbf{k}\cdot\mathbf{r}} \right]$$

$$\omega(\mathbf{k}) = |\mathbf{k}|,\ \ \varepsilon_1 \cdot \varepsilon_2 = 0,\ \ \varepsilon_s \cdot \mathbf{k} = 0$$

Quantization

$$a_s^+(\mathbf{k}) \text{ and } a_s(\mathbf{k}) \longrightarrow \text{ operators}$$

with commutation relations

$$\left[a_s(\mathbf{k}), a_{s'}^+(\mathbf{k}') \right] = \delta_{ss'} \delta^3(\mathbf{k} - \mathbf{k}')$$

Basic remarks on spatiotemporal dependence

Classically

Maxwell's equations (and boundary conditions) govern

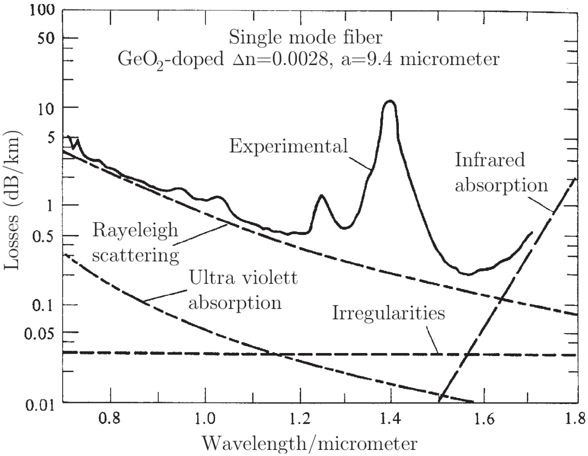
Quantum

Classical behaviour is inherited to correlations. Modal and material dispersion effects can be present in fibres and hollow waveguides.

Strategy

Analyze the simple classical ED first and apply the results to QED.

Losses for a single mode fibre



From Karlsson & Kristensson (1996-2011)

Dispersion

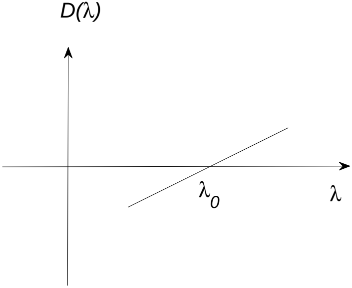
Dispersion shift

The diameter of the fibre is chosen to cancel the dispersion D at the design wavelength λ_0

$$D(\lambda_0) = D_m(\lambda_0) + D_w(\lambda_0) = 0$$

There is a remaining dispersion after the dispersion shift.

$D_m\ (< 0)$	material dispersion
$D_w\ (> 0)$	waveguide dispersion
$D = \dfrac{\mathrm{d}\tau_g}{\mathrm{d}\lambda}$	dispersion parameter
$\tau_g = \dfrac{\mathrm{d}k_z}{\mathrm{d}\omega}$	inverted group velocity
k_z	axial wave number
λ	wavelength in vacuum



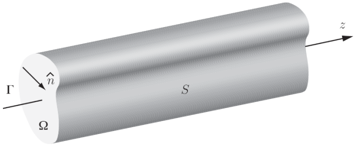
State of the art for fibres

Current bit rates for fibres

- 10 Gbit/s per channel in commercial systems
- 40 Gbit/s with wavelength dispersion multiplexing
- 1 Tbit/s in laboratories

From Karlsson & Kristensson (1996-2011)

TE/TM theory



The electromagnetic field is derived from two scalar potentials H_z and E_z:

$$\mathbf{H} = \mathbf{H}(E_z, H_z), \ \mathbf{E} = \mathbf{E}(E_z, H_z)$$

- TE waves $E_z = 0$ and $\varphi_{\text{TE}} = H_z$ as potential
- TM waves $H_z = 0$ and $\varphi_{\text{TM}} = E_z$ as potential

The TM and TE waves do not couple for a hollow waveguide with perfectly conducting walls.

The TM solution

$$E_z(x,y,z,t) = \sum_{n=1}^{\infty} v_n(x,y)\varphi_n(z,t).$$

$\varphi_n(z,t)$ is a solution to the Klein-Gordon equation

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial z^2} + m_n^2\right)\varphi_n(z,t) = 0.$$

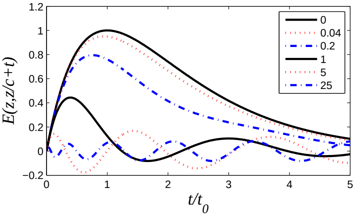
The complete set v_n of basis functions is a solution to

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + m_n^2\right)v_n(x,y) = 0, \qquad (x,y) \in \Omega, \quad v_n|_{\partial\Omega} = 0$$

$$\varphi_n(z,t) = \int \frac{dk}{2\sqrt{2\pi\omega_n(k)}} \left[a_n^+(k)e^{i\omega_n(k)t-ikz} + a_n(k)e^{-i\omega_n(k)t+ikz}\right]$$

$\omega_n(k) = \sqrt{k^2 + m_n^2}$.
Quantize by taking $a_n(k)$ and $a_n^+(k)$ as the annihilation and creation operators.

Dispersive behaviour of a pulse



The propagation of a double exponential pulse with $\bar{m} = mt_0 = 2$ and $\bar{z} = z/t_0$ as a parameter. The time t is taken after the arrival of the pulse. t_0 is the time at the maximum of the initial pulse.

Wave guide dispersion

- widens the pulse reducing the bit rate due to mixing of pulses.
- attenuates the peak level of the pulse obstructing observations

Dispersion for large distances

$$\widehat{E}(z,s) = \int_0^\infty E(z,t)\ \mathrm{e}^{-st}\mathrm{d}t.$$

The stationary phase method provides

$$\begin{array}{c} E(z,z+t) \sim_{z\to\infty} \sqrt{\dfrac{m_n}{\pi z}} \left[\dfrac{t}{z}(2+\dfrac{t}{z})\right]^{-3/4} \times \\ \mathrm{Re}\left\{\widehat{E}(0,-\mathrm{i}\omega_s)\mathrm{e}^{-\mathrm{i}m_n\left[\frac{t}{z}(2+\frac{t}{z})\right]^{1/2} z - \mathrm{i}\pi/4}\right\} \mathrm{u}(t) \end{array}, \qquad (1)$$

where $\omega_s = m_n(1+t/z)/\sqrt{t/z(2+t/z)}$.

- A rigorous method with error bounds, Olver (1974).
- A valuable analytical tool for analyzing correlations at large distances.

High accuracy of asymptotic formulae

Comparison between asymptotic formulae (long) and an independent numerical method (short)

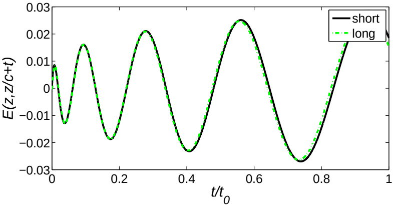


Figure: $\bar{z} = z/t_0 = 100, \bar{m} = mt_0 = 2$

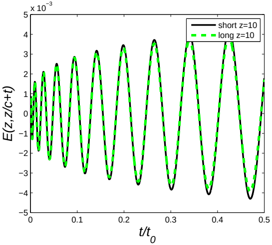


Figure: $\bar{z} = z/t_0 = 10, \bar{m} = mt_0 = 20$

A high accuracy is demonstrated for the asymptotic formula at large distances. The accuracy is even better for larger times.

Problem formulation

Determine the spatiotemporal dependence of correlation functions for entangled photons in waveguides

- Only waveguide dispersion.
- The hollow waveguide model with frequency independent mass is used.
- This will demonstrate the effect of dispersion but it is exaggerated.

Probability density

We consider only the $z-$component of the TM-field that solves the Klein-Gordon equation. We define a one particle state

$$|\psi_1\rangle = \int g(k) a_k^\dagger |0\rangle,$$

where $|0\rangle$ is the Fock vacuum. The probability density to detect the photon at the point z along the waveguide at time t is proportional to

$$P(z,t) = \langle \psi_1 | \varphi^{(-)}(z,t) \varphi^{(+)}(z,t) | \psi_1 \rangle,$$

where

$$\varphi^{(+)}(z,t) = \frac{1}{(2\pi)^{1/2}} \int_{\mathbb{R}} \frac{dk}{\sqrt{2\omega_k}} a_k e^{-i\omega_k t + ikz}, \varphi^{(-)}(z,t) = h.c.,$$

and $\omega_k = \sqrt{k^2 + m^2}$, $m > 0$.

Probability density II

The probability can be written as

$$P(z,t) = |A(z,t)|^2.$$

Like the classical case, $A(z,t)$ is a solution to the Klein-Gordon equation and

$$A(z,t) = \langle 0|\varphi^{(+)}(z,t)|\psi_1\rangle = \int dk g(k) \frac{e^{-i\omega_k t + ikz}}{2\sqrt{2\pi\omega_k}}.$$

For sufficiently smooth function $g(k)$ one can use the stationary phase method to get for the probability

$$P(z,t) \sim_{z\to\infty} \frac{1}{4t} \frac{\left| g\left(m\frac{z}{t} \Big/ \sqrt{1-\frac{z^2}{t^2}} \right) \right|^2}{1-\frac{z^2}{t^2}}$$

Correlation function for the biphoton

Now we define a two particle entangled state (biphoton)

$$|\psi\rangle = \int f(k_1, k_2) a^\dagger_{k_1} a^\dagger_{k_2} |0\rangle,$$

where $|0\rangle$ is the Fock vacuum and $f(k_1, k_2) = f(k_2, k_1)$ (bosons). The probability to detect one particle at the point z_1 along the waveguide at time t_1 and another particle at the space point z_2 at time t_2 is proportional to

$$P(z_1, t_1, z_2, t_2) = \langle\psi|\varphi^{(-)}(z_1, t_1)\varphi^{(-)}(z_2, t_2)\varphi^{(+)}(z_2, t_2)\varphi^{(+)}(z_1, t_1)|\psi\rangle.$$

Correlation function for the biphoton II

The probability for finding two photons is proportional to

$$P(z_1, t_1, z_2, t_2) = |A(z_1, t_1, z_2, t_2)|^2,$$

where

$$A(z_1, t_1, z_2, t_2) = \langle 0|\varphi^{(+)}(z_1, t_1)\varphi^{(+)}(z_2, t_2)|\psi\rangle = \int dk_1 dk_2$$

$$\left\{ \frac{e^{-i\omega_{k_2} t_2 + ik_2 z_2}}{2\sqrt{2\pi\omega_{k_2}}} \frac{e^{-i\omega_{k_1} t_1 + ik_1 z_1}}{2\sqrt{2\pi\omega_{k_1}}} f(k_1, k_2) + (k_1 \leftrightarrow k_2) \right\}.$$

Correlation function for the biphoton III

We get for the probability of finding two photons:

$$P(z_1, t_1, z_2, t_2) \sim_{z_1, z_2 \to \infty} \frac{1}{16 t_1 t_2} \frac{\left| f\left(k_{10}, k_{20}\right) \right|^2}{\left(1 - \frac{z_1^2}{t_1^2}\right)\left(1 - \frac{z_2^2}{t_2^2}\right)}$$

$$\left| e^{-it_1(\omega_{k_{10}} - ik_{10} z_1)/t_1} e^{-it_2(\omega_{k_{02}} - ik_{20} z_2/t_2)} + (k_1 \leftrightarrow k_2) \right|^2$$

where

$$k_{i0} = m \frac{z_i}{t_i} \Bigg/ \sqrt{1 - \frac{z_i^2}{t_i^2}}$$

$$\omega_{k_{i0}} = m \Bigg/ \sqrt{1 - \frac{z_i^2}{t_i^2}}$$

Dispersion is causing spreading as well as attenuation of the maximum amplitude.

Correlation function for the biphoton IV

Without using asymptotic methods it can be shown that there exists a constant C such that

$$P(z_1, t_1, z_2, t_2) \leq \frac{C}{|t_1||t_2|},$$

for all z_1, t_1, z_2, t_2 (attenuation of the maximum amplitude).
An explicit expression in a special case is given by Yang et al (2008):

$$f(k_1, k_2) = \frac{i}{k_0^2} f_P(k_1 + k_2)\sqrt{6k_1 k_2(k_1 + k_2)},$$

$f_P(k_1 + k_2)$ is a Gaussian function describing the pumping photons.

Entanglement over long distances but dispersion is a big problem

Our predictions on dispersion for the hollow waveguide are seen in the optic fibre:

100-km entanglement distribution, Chang et al 2008

- Means to reduce unwanted effects from dispersion
  - Dispersion shift
  - A very narrow band pass filter in the fibre: 0.8 nm (the wavelength for the pumping laser is 15559 nm)
- Still the time duration of the photon pair increased from 4 ps to 25 ps by the 100 km fibre
- Super conducting detectors are required for observing the photons.

Theory for photon correlations in an optic fibre

Classical hollow waveguide with material dispersion and dispersion shift
The analysis is formally the same as above but larger distances are required until algebraic decay is prominent.

Classical fibre
More complicated. Non-discrete modes appear but decay more quickly than the lowest discrete mode. One discrete mode with a band limit source is a good model.

Quantum fibre
One discrete mode with a band limit source should be a good model. The quantization problem for a mode with complex valued and frequency dependent mass remains.

Summary

The spatiotemporal dependence of correlation functions for entangled photons in waveguides

- Accurate asymptotic expressions for photon correlation functions in hollow waveguides
- Photon correlations spread in space and time due to dispersion
- Qualitative agreement with experiments on 100-km fibres
- Discussion of theory for optic fibres